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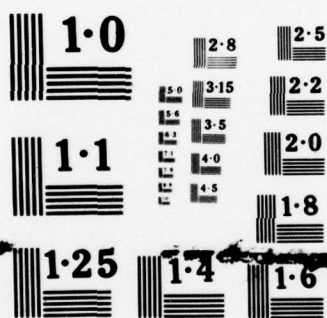
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LARGE DEVIATIONS FOR DIFFUSIONS DEPENDING
ON SMALL PARAMETERS: A STOCHASTIC CONTROL METHOD

by

Wendell H. Fleming

This paper summarizes a stochastic control method to derive estimates of Ventcel-Freidlin type that a solution trajectory of a stochastic differential equation hits a given set B during a given time interval. A more detailed treatment of the method is given in the author's paper, "Exit probabilities and optimal stochastic control" to appear in Applied Mathematics and Optimization.

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LARGE DEVIATIONS FOR DIFFUSIONS DEPENDING
ON SMALL PARAMETERS: A STOCHASTIC CONTROL METHOD

Wendell H. Fleming⁺

ABSTRACT

This paper considers stochastic differential equations depending on a small parameter ϵ , which enters as a coefficient in the noise term of the equation. The Ventcel-Freidlin estimates give an asymptotic formula as $\epsilon \rightarrow 0$ for the probability that a solution trajectory hits a given compact set B during a given time interval. A stochastic control method to obtain such estimates is outlined. Detailed proofs are given elsewhere.

1. The problem. Consider a Markov diffusion process ξ^ϵ on n -dimensional R^n which obeys the stochastic differential equations

$$(1.1) \quad d\xi^\epsilon = b[t, \xi^\epsilon(t)]dt + \sqrt{\epsilon} \sigma[t, \xi^\epsilon(t)]dw, \quad s \leq t,$$

with initial data $\xi^\epsilon(s) = x$. Here w is an n -dimensional brownian motion and ϵ a positive parameter. Let B be a compact set, $B \subset R^n$, and let τ_B^ϵ denote the first time t such that $\xi^\epsilon(t) \in B$. If $\xi^\epsilon(t) \in R^n - B$ for all $t \geq s$ we set $\tau_B^\epsilon = +\infty$. Given $T > s$, let

$$(1.2) \quad q_B^\epsilon = P(\tau_B^\epsilon \leq T).$$

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Let us also consider the unperturbed system

$$(1.3) \quad d\xi^0 = b[t, \xi^0(t)]dt, \quad s \leq t,$$

with the same initial data $\xi^0(s) = x$. If $\tau_B^0 > T$, then $q_B^\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. The Ventcel-Freidlin estimates [3], [5] give a more precise statement about the rate of convergence to 0 of q_B^ε , under certain assumptions about the coefficients b, σ in (1.1) and the set B . A different method, based on stochastic control ideas, was used in [1], [2] to obtain estimates of Ventcel-Freidlin type. In this note we outline this stochastic control method, as it applies to the problem of estimating q_B^ε .

2. Upper and lower estimates. Let us assume that b is Lipschitz on R^{n+1} , and that σ together with its inverse σ^{-1} are bounded and Lipschitz. Some results in which σ degenerates in a particular way appear in [4]. Let $a = \sigma\sigma'$, and consider the variational integrand

$$(2.1) \quad L(t, \phi, \dot{\phi}) = \frac{1}{2} (b(t, \phi) - \dot{\phi})' a^{-1}(t, \phi) (b(t, \phi) - \dot{\phi}).$$

Let

$$\Gamma_B = \{\phi \in C([s, T]; R^n) : \phi(s) = x, \exists \theta \in [s, T] \ni \phi(\theta) \in B\},$$

$$(2.2) \quad I_B = \min_{\Gamma_B} \int_s^T L[t, \phi(t), \dot{\phi}(t)] dt.$$

(The integral on the right side is defined to be $+\infty$ if ϕ is not absolutely continuous.) Let

$$(2.3) \quad I_B^\varepsilon = -\varepsilon \log q_B^\varepsilon.$$

Theorem 1. $I_B \leq \liminf_{\varepsilon \rightarrow 0} I_B^\varepsilon.$

This theorem gives an upper estimate for q_B^ε . We shall outline a stochastic control proof in §3, and refer to [2] for details.

In order to state a lower estimate, we make the following additional assumption: There exists a relatively open set $\Gamma_B^0 \subset \Gamma_B$ such that

$$(2.4) \quad \inf_{\Gamma_B^0} \int_s^T L[t, \phi(t), \dot{\phi}(t)] dt = I_B.$$

By relatively open we mean that given $\phi \in \Gamma_B^0$ there exists $\delta > 0$ such that $\psi \in C([s, T]; \mathbb{R}^n)$, $\psi(s) = x$ and $\|\psi - \phi\| < \delta$ imply $\psi \in \Gamma_B^0$.

Theorem 2. If (2.4) holds, then $I_B \geq \limsup_{\epsilon \rightarrow 0} I_B^\epsilon$.

Theorem 2 is an immediate consequence of the first Ventcel-Freidlin estimate [3, p. 332]. That estimate is rather easily obtained from the Girsanov formula. On the other hand, the usual proof of Theorem 1 is based on a somewhat more technically complicated second Ventcel-Freidlin estimate [3, p. 334]. Our method furnishes an alternative proof, as well as a different intuition for arriving at such results.

From Theorems 1 and 2 one has $I_B^\epsilon \rightarrow I_B$ as $\epsilon \rightarrow 0$, which is the desired result about the probability of large deviations.

3. Stochastic control method. Let us fix $T > 0$ and consider $0 < s < T$. Let $Q = (0, T) \times (\mathbb{R}^n - B)$, and write $q_B^\epsilon = q_B^\epsilon(s, x)$. Then $q_B^\epsilon \in C^{1,2}(Q)$ and satisfies in Q the backward partial differential equation

$$(3.1) \quad \frac{\partial q_B^\epsilon}{\partial s} + \frac{\epsilon}{2} \sum_{i,j=1}^n a_{ij}(s, x) \frac{\partial^2 q_B^\epsilon}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(s, x) \frac{\partial q_B^\epsilon}{\partial x_i} = 0.$$

The function $I_B^\epsilon = -\epsilon \log q_B^\epsilon$ satisfies the nonlinear equation

$$(3.2) \quad \frac{\partial I_B^\epsilon}{\partial s} + \frac{\epsilon}{2} \sum_{i,j=1}^n a_{ij}(s, x) \frac{\partial^2 I_B^\epsilon}{\partial x_i \partial x_j} + H(s, x, \nabla \phi_B^\epsilon) = 0,$$

where for each row vector p

$$(3.3) \quad H(s, x, p) = -\frac{1}{2} p a(s, x) p' + p \cdot b(s, x).$$

The function $H(s, x, \cdot)$ is dual to $L(s, x, \cdot)$, in the sense of duality for concave and convex functions. If one puts formally $\epsilon = 0$ in (3.2), one gets the Hamilton-Jacobi equation associated with the variational integrand L . The fact that Γ_B includes only curves ϕ which reach B by time T corresponds formally to the condition $q_B^\epsilon(T, x) = 0$ for $x \notin B$, and hence $I_B^\epsilon(T, x) = +\infty$. The connection with the Hamilton-Jacobi equation indicates,

but of course does not prove, results of the kind stated in §2.

Lemma. Let $B = \partial D$, where $D \subset \mathbb{R}^n$ is open, bounded with C^2 boundary ∂D . Then $I_B \leq \liminf_{\epsilon \rightarrow 0} I_B^\epsilon$, for any $(s, x) \in (0, T) \times D$.

This result is proved in [2, §7], by an argument which we sketch below. From the Lemma, Theorem 1 is obtained as follows. There exist $D_1 \subset D_2 \subset \dots$, with union $\mathbb{R}^n - B$, such that $\partial D_n = \tilde{B}_n \cup \{|x| = n\}$, with \tilde{B}_n of class C^2 and \tilde{B}_n contained in the n^{-1} -neighborhood of B . We take $B_n = \partial D_n$ and note that $q_B^\epsilon \leq q_{B_n}^\epsilon$,

$$\liminf_{\epsilon \rightarrow 0} I_B^\epsilon \geq \liminf_{\epsilon \rightarrow 0} I_{B_n}^\epsilon \geq I_{B_n}.$$

Moreover, $I_{B_n} \rightarrow I_B$ as $n \rightarrow \infty$.

The proof in [2, §7] of the Lemma is based on a "penalty" function method. Let $\phi(x)$ be Lipschitz, with $\phi(x) > 0$ for $x \notin \partial D$, $\phi(x) = 0$ for $x \in \partial D$. Let $J^\epsilon(s, x)$ be the solution of (3.2) in $(0, T) \times D$, with $J^\epsilon(s, x) = 0$ for $0 < s < T$, $x \in \partial D$, $J^\epsilon(T, x) = \phi(x)$. Then $J^\epsilon(s, x)$ satisfies the dynamic programming equation for an optimal stochastic control problem in which the drift term $b[t, \xi^\epsilon(t)]$ in (1.1) is replaced by an arbitrary, bounded, nonanticipative control process $v(t)$. Instead of (1.1), one now has the stochastic differential equation

$$d\eta = v(t)dt + \sqrt{\epsilon} \sigma[t, \eta(t)]dw.$$

The quantity to be minimized is

$$E\left\{\int_s^\theta L(t, \eta(t), v(t))dt + \phi[\theta, \eta(\theta)]\right\},$$

where $\theta = \min(T, \text{exit time from } D \text{ of } \eta(t))$. The minimum of this expression is $J^\epsilon(s, x)$. It is shown in [2] that $\liminf_{\epsilon \rightarrow 0} J^\epsilon \geq J$, where $J(s, x)$

is the minimum of $\int_s^\theta L[t, \phi(t), \dot{\phi}(t)]dt + \phi[\theta, \phi(\theta)]$ taken among all (ϕ, θ)

with $\phi(s) = x$ and either $\theta = T$ or $\theta < T$, $\phi(\theta) \in \partial D$. For $M = 1, 2, \dots$ we now take $\phi = \phi_M = M\psi$ for fixed ψ , and write $J^\epsilon = J^{\epsilon M}$, $J = J^M$. It is easy to show that $I \leq \liminf_{M \rightarrow \infty} J^M$. Moreover, $J^{\epsilon M} \leq I^\epsilon$. Then $J^M \leq \liminf_{\epsilon \rightarrow 0} I_M^\epsilon$

for each M , from which the Lemma follows.

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